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On the structure of the sandpile identity element on Sierpiński gasket graphs

Robin Kaiser[1] · **Ecaterina Sava-Huss**[2] · **Julia Überbacher**[2]

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Abstract
We consider the identity of the abelian sandpile group of finite approximation graphs of the Sierpiński gasket, and we show that the second-order term in the scaling limit converges to the path distance to the nearest corner on the Sierpiński gasket. The proof relies on a decomposition of the identity of the sandpile group into the sum of a constant function and the Laplacian of the graph distance on the approximating graphs.

Keywords Abelian sandpiles · Sandpile group · Identity sandpile · Sierpiński gasket · Green's function · Scaling limit · Second-order limit

Mathematics Subject Classification 31E05 · 60J10 · 60J45 · 05C81

1 Introduction

Bak et al. (1987, 1988) introduced the sandpile automaton on lattices as a system that displays self-organized criticality: it evolves into a critical state without the fine-tuning of external parameters. Dhar (1990) generalized this model to arbitrary finite graphs and noticed that the dynamics of the model are abelian, which led him to refer to it as the abelian sandpile model. Because of its relevance in physics and its interesting connections to many different fields in mathematics, the abelian sandpile model has since attracted the interest of physicists and mathematicians alike.

For a finite graph $G = (V \cup \{s\}, E)$ with a designated sink vertex s, a sandpile configuration is a collection of indistinguishable particles on the vertices of G. We call a vertex stable, if the number of particles is smaller than its degree, and unstable otherwise. The unstable vertices

✉ Robin Kaiser
ro.kaiser@tum.de

Ecaterina Sava-Huss
Ecaterina.Sava-Huss@uibk.ac.at

Julia Überbacher
Julia.Ueberbacher@uibk.ac.at

[1] Departement of Mathematics, CIT, Technische Universität München, Boltzmannstr. 3, 85748 Garching bei München, Germany

[2] Universität Innsbruck, Institut für Mathematik, Technikerstraße 13, 6020 Innsbruck, Austria

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can be legally toppled by sending one particle to each neighboring vertex. During a toppling, if particles are sent to the sink, they are lost forever. By using these toppling dynamics, one can define the abelian sandpile Markov chain on the space of stable sandpile configurations in the following manner. At each time step, a particle is added at a uniformly random location, and subsequently, the configuration is stabilized by performing all possible legal topplings.

Research on the abelian sandpile model spans topics ranging from the limit shape in the single-source model—where all particles are initially placed on a single vertex and the stabilized configuration is analyzed (Pegden and Smart 2013; Levine et al. 2016; Fey et al. 2010)—through the dynamics of the sandpile Markov chain, including its mixing behaviour and stationary distribution (Jerison et al. 2019; Hough et al. 2019; Járai and Lyons 2007), to the infinite-volume limit of the stationary distribution and the size of avalanches at criticality (Járai and Werning 2014; Athreya and Járai 2004; Bhupatiraju et al. 2017; Eckmann et al. 2023).

The set of recurrent configurations for the sandpile Markov chain forms an abelian group—also known as the sandpile group or critical group—which is the focus of this note. In particular, we study the identity element of the sandpile group (shortly *sandpile identity*) on Sierpiński gasket approximation graphs. The sandpile identity has been examined on various graphs; for example, Le Borgne and Rossin (2002) analyzes it on rectangular grids under certain size constraints, and Melchionna (2022) investigates the identity on elliptical subsets of the square lattice. Notably, the conjecture of Dhar et al. (1995) concerning a central square of vertices with two particles each in the identity on square boxes in $\mathbb{Z}^2$ remains open.

On the Sierpiński gasket, recent work has characterized and analyzed the identity element of the sandpile group. In particular, Chen and Kudler-Flam (2020) describes its structure on Sierpiński gasket graphs under various sinked boundary conditions, where one or more corner vertices are the designated sink vertices and Kaiser et al. (2024) gives a recursive characterization of the identity on the approximating graphs for normal boundary conditions, where each of the three corner vertices is connected by two edges to the sink vertex. In the current paper, we focus on the scaling limit of the sandpile identity on the approximation graphs of the Sierpiński gasket, previously studied in Kaiser and Sava-Huss (2026), where it is shown that the piecewise-constant extensions of the identities converge in the weak* sense to a constant function on the gasket. However, the structure of the identity is lost under the weak* topology. Here, we propose a different notion of limit by convolving the sandpile identity with the stopped Green's function. This approach retains the identity's structure and addresses a question raised in Kaiser and Sava-Huss (2026). We now present our main result. Formal definitions of the objects in the statement are deferred to Sect. 2.

Theorem 1.1 *Let id_n be the identity of the sandpile group of the n-th approximation graph SG_n of the Sierpiński gasket SG, for $n \in \mathbb{N}$. Denote by g_n the Green function of the simple random walk on SG_n stopped at the corners, and let G be the Green function with Dirichlet boundary conditions on the corner vertices on the Sierpiński gasket SG.*

(I1) The function $g_n * \mathsf{id}_n$ grows asymptotically as 5^n, and for all $x \in SG$ we have

$$\lim_{n\to\infty} \frac{1}{5^n}(g_n * \mathsf{id}_n)^{\blacktriangle}(x) = 4\int_{SG} G(x,y)d\mu(y).$$

(I2) The second order term of $g_n * \mathsf{id}_n$ is given by the graph distance to the corners of SG_n

$$\lim_{n\to\infty} \frac{1}{2^n}\left(g_n * \mathsf{id}_n - \frac{8}{3}\sum_{y\in V_{SG_n}} g_n(y,\cdot)\right)^{\blacktriangle}(x) = -\frac{1}{3}\mathsf{d}(x),$$

where $\mathsf{d}(x)$ *is the shortest path distance from* x *to the three corners in* SG.

(I3) For all post-critically finite points $x \in V_*$ *it holds that*

$$\lim_{n\to\infty} \frac{1}{2^n}\left(g_n * \mathsf{id}_n(x) - 4\cdot 5^n \int_{SG} G(x,y)d\mu(y)\right) = -\frac{1}{3}\mathsf{d}(x).$$

Before proving Theorem 1.1, we give an informal and intuitive description of the different scaling factors and objects appearing in the statement:

- The factor 5^n represents the time acceleration factor required for random walks on SG_n to converge to a continuous Brownian motion on the Sierpinski gasket SG.
- The constant $8/3$ corresponds to the asymptotic average chip density of the sandpile identity element id_n as $n \to \infty$; see Fig. 3.
- The scaling constant 4 emerges naturally from the standard convention used to relate the discrete and continuum Laplacians (and their inverse Green's functions) on SG.
- The measure μ denotes the standard self-similar probability measure on SG.

The proof of Theorem 1.1 relies on decomposing the sandpile identity into the distance to the nearest corner vertex and a function with constant Laplacian in the interior of the gasket. More precisely, after convolving the sandpile identity with the stopped Green function on SG_n, we obtain $g_n * \mathsf{id}_n = \frac{8}{3}h_n - \frac{1}{3}d_n$, where d_n denotes the distance to the closest corner, and h_n is the function with Laplacian equal to 1 at every non-corner vertex of SG_n and value 0 at the corners. Using results from analysis on fractals, we then deduce the limits stated in Theorem 1.1.

Outline. Section 2 introduces the abelian sandpile model, the Sierpiński gasket SG and its approximation graphs SG_n, and the necessary tools from analysis on fractals used in the proof of Theorem 1.1. In Sect. 3, we establish the key ingredient: a decomposition of the sandpile identity on the approximation graphs into a function with constant Laplacian and the distance to the nearest corner. Finally, in Sect. 4, we prove Theorem 1.1: Proposition 4.3 is (I1), Proposition 4.1 is (I2), and Proposition 4.2 is (I3).

2 Preliminaries

Abelian sandpiles. For a comprehensive overview on the abelian sandpile model, see the survey Járai (2018). Let $G = (V \cup \{s\}, E)$ be a finite, connected, undirected graph with a distinguished vertex s (the sink). For $v \in V \cup \{s\}$, let $\deg_G(v)$ denote its degree in G, i.e., the number of its neighbors; when the graph is clear from context, we write $\deg(v)$. For vertices $v, w \in V$, we write $v \sim w$ if they are adjacent in G (connected by an edge).

A sandpile on G is a function $\sigma : V \to \mathbb{N}$ that assigns to each vertex $v \in V$ the number of particles at v. A vertex v is unstable if $\sigma(v) \geq \deg(v)$; otherwise it is stable. The configuration σ is called stable if it is stable at every vertex. For each vertex $v \in V$, define a toppling T_v at v by removing $\deg(v)$ particles from v and sending one particle along each incident edge. Writing a_{vw} for the number of edges between v and w, the toppling redistributes particles as follows:

$$(T_v\sigma)(u) = \begin{cases} \sigma(u) - \deg(v), & u = v, \\ \sigma(u) + a_{vu}, & u \sim v, \\ \sigma(u), & \text{otherwise.} \end{cases}$$

A toppling is legal if it results in a configuration with all nonnegative entries, i.e. if the toppled vertex v is unstable in σ. Any particles sent to the sink are permanently removed

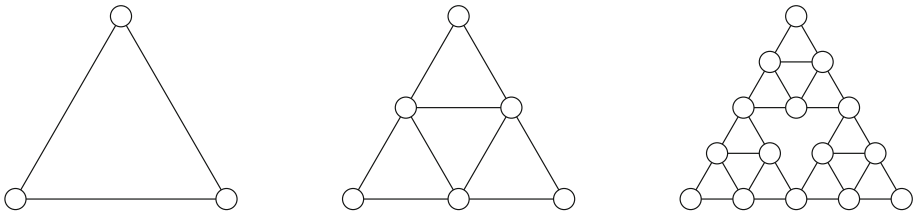


Fig. 1 The first three Sierpiński gasket approximation graphs SG_0, SG_1 and SG_2

from the system. We view σ as a vector indexed by V, with each entry recording the number of particles at the corresponding vertex. With this identification, toppling operations can be expressed using the graph Laplacian of G. The Laplacian $\Delta \in \mathrm{Mat}_{|V\cup\{s\}|\times|V\cup\{s\}|}(\mathbb{Z})$ is the matrix with entries

$$\Delta_{vw} = \begin{cases} -a_{vw}, & v \neq w,\ v \sim w, \\ \deg(v), & v = w, \\ 0, & \text{otherwise.} \end{cases}$$

Let Δ' denote the reduced Laplacian of G with respect to s, obtained by restricting Δ to $V \times V$ (i.e., deleting the row and column corresponding to the sink). Then a toppling at v is given by

$$T_v\sigma = \sigma - \Delta'\delta_v,$$

where δ_v is the vector with a 1 at index v and 0 elsewhere. This formulation immediately shows that topplings commute (the model is abelian):

$$T_vT_w\sigma = T_wT_v\sigma \quad \text{for all } v, w \in V.$$

The stabilization σ° of a sandpile σ is the unique stable configuration obtained by performing all legal topplings. This is well defined: there exists a sequence of vertices $v_1, \ldots, v_n$—unique up to permutation—such that all topplings are legal and the resulting configuration $\sigma^\circ = T_{v_n} \cdots T_{v_1}\sigma$ is stable. See Járai (2018) for details.

Sandpile Markov chain and the sandpile group. Given sandpiles and stabilization, one defines a Markov chain on the set of stable configurations of G, known as the abelian sandpile Markov chain, as follows. Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. uniformly distributed on V, and start from any stable configuration σ_0. Define the Markov chain $(\sigma_n)_{n\in\mathbb{N}}$ by

$$\sigma_{n+1} = (\sigma_n + \delta_{X_n})^\circ,$$

i.e., at time-step n, given σ_n, we add one particle to a uniformly chosen vertex in G and then we stabilize $\sigma_n + \delta_{X_n}$. The chain $(\sigma_n)_{n\in\mathbb{N}}$ is called *the abelian sandpile Markov chain*, and its set of recurrent states forms an abelian group denoted $\mathcal{R}_G$, where the group operation $\oplus : \mathcal{R}_G \times \mathcal{R}_G \to \mathcal{R}_G$ is the pointwise addition followed by stabilization $\sigma \oplus \omega = (\sigma + \omega)^\circ$. Since $\mathcal{R}_G$ is an abelian group, it has a neutral element under $\oplus$: a recurrent configuration id_G, called the sandpile identity of G.

The Sierpiński gasket. We define Sierpiński gasket approximation graphs SG_n, for $n \in \mathbb{N}$ as in Barlow and Perkins (1988). For $x \in \mathbb{R}^2$ and a graph $G = (V, E)$ that can be embedded in $\mathbb{R}^2$, we denote by $x + G$ the graph

$$x + G = (\{x + v : v \in V\},\ \{(x + a, x + b) : (a, b) \in E\}).$$

For $a \in \mathbb{R}$, we write aG for the graph

$$aG := (\{av : v \in V\}, \{(av, aw) : (v, w) \in E\}).$$

Let $u_1 = (0, 0)$, $u_2 = \frac{1}{2}(1, \sqrt{3})$ and $u_3 = (1, 0)$. We define $SG_0 = (V_{SG_0}, E_{SG_0})$ to be the complete graph with vertex and edge set given by

$$V_{SG_0} = \{u_1, u_2, u_3\} \quad \text{and} \quad E_{SG_0} = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_1, u_3\}\}.$$

Recursively, for $n \geq 1$ the n-th Sierpiński gasket approximation graph $SG_n = (V_{SG_n}, E_{SG_n})$ is

$$SG_n = \frac{1}{2}\big(SG_{n-1} \cup (u_2 + SG_{n-1}) \cup (u_3 + SG_{n-1})\big).$$

Throughout this paper, we will also use the less cumbersome notation V_n for the set of vertices in the n-th iteration. Notice that SG_n is obtained from SG_{n-1} by copying SG_{n-1} three times and shifting one copy to the right and one copy diagonally upwards, and then rescaling the union of the three copies. Figure 1 displays the first three Sierpiński gasket graphs SG_0, SG_1 and SG_2. Define the union of the vertices of all the approximation graphs

$$V_* = \bigcup_{n \in \mathbb{N}} V_n.$$

The Sierpiński gasket is defined as the closure of V_* in $\mathbb{R}^2$.

2.1 Potential theory on the Sierpiński gasket

This subsection is devoted to introducing the necessary tools and framework from analysis on fractals—specifically analysis on the Sierpiński gasket SG—needed to prove Theorem 1.1. For a comprehensive overview, see Kigami (2001) and Strichartz (2006).

Standard self-similar measure. For each $i \in \{1, 2, 3\}$, define $\psi_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\psi_i(x) = \tfrac{1}{2}(x - u_i) + u_i,$$

where u_1, u_2, u_3 are the vertices of the previously defined equilateral triangle. For a word $w = w_1 w_2 \dots w_n \in \{1, 2, 3\}^n$ of length $n \in \mathbb{N}$, we write $\psi_w = \psi_{w_n} \circ \cdots \circ \psi_{w_1}$. We take the σ-algebra of measurable sets on SG to be generated by sets of the form $\{\psi_w(SG) : w \in \bigcup_{m \geq 1} \{1, 2, 3\}^m\}$. The standard self-similar measure μ is the unique measure with $\mu(\psi_w(SG)) = 3^{-|w|}$, where $|w|$ is the length of the word w. Notice that this standard self-similar measure is up to scalar multiples a $\log_2(3)$-dimensional Hausdorff measure on the Sierpiński gasket with respect to the Euclidean metric. See also Chapter 1.2 in Strichartz (2006, pp. 5–9) for more details.

Green function on SG_n. Let $(X_t^n)_{t \in \mathbb{N}}$ denote the simple random walk on SG_n, i.e., the Markov chain with transition probabilities given by

$$p(x, y) = \begin{cases} \frac{1}{4}, & \text{if } x \underset{n}{\sim} y,\ x \neq u_i \\ \frac{1}{2}, & \text{if } x \underset{n}{\sim} y,\ x = u_i \end{cases}$$

for $i \in \{1, 2, 3\}$, where $x \underset{n}{\sim} y$ means that $x, y \in V_{SG_n}$ are neighbours in SG_n. For $x \in V_{SG_n}$, denote by $\mathbb{E}_x$ the expectation with respect to the probability measure $\mathbb{P}(\cdot | X_0^n = x)$. The

Green function $g_n(x, y)$ on SG_n is the expected number of visits to y by the walk started at x before it hits one of the three corner vertices u_1, u_2, u_3:

$$g_n(x, y) = \mathbb{E}_x\left[\sum_{t=0}^{\tau_n - 1} \mathbf{1}_{\{X_t^n = y\}}\right], \quad \text{where } \tau_n = \inf\left\{t \geq 0 : X_t^n \in \{u_1, u_2, u_3\}\right\}.$$

If we write Δ_n for the graph Laplacian on SG_n, then it holds that

$$\Delta_n g_n(x, \cdot)(y) = \delta_x(y), \quad \text{for all} \quad x, y \in SG_n \backslash \{u_1, u_2, u_3\}.$$

Piecewise constant continuation to SG. For $x \in SG$, define $x_n \in V_{SG_n}$ as $x_n = \operatorname{argmin}\{d(x, y) : y \in SG_n\}$. For uniqueness, if there are several minimizers, choose the one furthest to the right. For a function $f : V_{SG_n} \to \mathbb{R}$ we define the piecewise constant continuation of f to SG by $f^{\blacktriangle} : SG \to \mathbb{R}$, $f^{\blacktriangle}(x) = f(x_n)$.

Laplacian operator on SG. The Laplacian operator on the Sierpiński gasket SG, denoted by Δ, can be defined via a weak formulation using an energy form on SG. For $n \in \mathbb{N}$, we define the energy form $\mathcal{E}_n$ on SG_n as

$$\mathcal{E}_n(f, g) = \left(\frac{5}{3}\right)^n \sum_{x \underset{n}{\sim} y} \big(f(x) - f(y)\big)\big(g(x) - g(y)\big),$$

where $f, g : SG \to \mathbb{R}$. Observe that the $5/3$ factor in the energy functional corresponds to the increase in minimal energy when a function on the n-th gasket is extended to the $(n + 1)$-st. For details, see Strichartz (2006, Section 1.3). Using these finite iteration energy forms $\mathcal{E}_n$ we define an energy form on SG via

$$\mathcal{E}(f, g) = \lim_{n \to \infty} \mathcal{E}_n(f|_{V_{SG_n}}, g|_{V_{SG_n}}).$$

Note that the limit is not necessarily finite. So, we also define the domain of the energy functional as

$$\operatorname{dom}(\mathcal{E}) := \{f : SG \to \mathbb{R} \mid \mathcal{E}(f, f) < \infty\}.$$

With this energy form, we define a weak Laplacian on SG. For $u \in \operatorname{dom}(\mathcal{E})$ and continuous $f : SG \to \mathbb{R}$, we write $u \in \operatorname{dom}(\Delta)$ and $\Delta u = f$ if and only if

$$\mathcal{E}(u, v) = -\int_{SG} f\, v \, d\mu \quad \text{for all } v \in \operatorname{dom}(\mathcal{E}).$$

Green function on SG. For Theorem 1.1, we must specify the function on SG that arises as the scaling limit of the discrete Green functions g_n defined above. Specifically, we define the Green function on the Sierpiński gasket SG, i.e., the integral kernel of the inverse Laplacian on SG. We define the Green function subject to Dirichlet boundary conditions on SG via harmonic splines as explained in Strichartz (2006, Section 2.6.) by

$$G(x, y) := \lim_{M \to \infty} \sum_{m=0}^{M} \sum_{z, z' \in V_{SG_{m+1}} \backslash V_{SG_m}} g(z, z')\, \Psi_z^{m+1}(x)\, \Psi_{z'}^{m+1}(y),$$

where

$$g(z, z') = \begin{cases} \dfrac{9}{50}\left(\dfrac{3}{5}\right)^m, & z = z' \text{ and } z \in V_{SG_{m+1}} \backslash V_{SG_m}, \\ \dfrac{3}{50}\left(\dfrac{3}{5}\right)^m, & z \neq z',\ z, z' \in V_{SG_{m+1}} \backslash V_{SG_m},\ \text{and } z, z' \in \psi_w(SG) \text{ for some } |w| = m, \end{cases}$$

and, for $z \in V_{SG_m}$, the level-m harmonic spline Ψ_z^m is the unique piecewise harmonic function on SG satisfying

$$\Psi_z^m(x) = \begin{cases} 1, & x = z, \\ 0, & x \in V_m \backslash \{z\}. \end{cases}$$

A key feature of the Green function on SG is that, up to the scaling factor $(3/5)^n$, it matches the stopped Green function g_n on the vertices of SG_n.

Lemma 2.1 *For all $n \in \mathbb{N}$ and all $x, y \in V_n$ it holds $\left(\frac{3}{5}\right)^n g_n(x, y) = G(x, y)$.*

For a proof see, for instance, Lemma 2.4 in Freiberg et al. (2024).

2.1.1 Auxiliary results

In the proofs of the scaling limit in Sect. 4, we need several technical lemmas. These include the mean-value property of harmonic functions on SG and properties of the Green function G on the SG, specifically the variation of the Green function within a given n-cell.

Lemma 2.2 *Let $h : SG \to \mathbb{R}$ be a harmonic function on $SG\backslash V_0$. Then for every $m, n \in \mathbb{N}$ with $m > n$ and $x \in V_n\backslash V_0$ it holds that*

$$\frac{1}{\mu(B(x, 2^{-m}))} \int_{B(x,2^{-m})} h(y) d\mu(y) = h(x).$$

Proof The statement follows from the well-known fact (Strichartz 2006, Section 1.3.) that a harmonic function h on an n-cell of SG is determined by its values at the three corner vertices via the 1/5–2/5 rule. More precisely, if $x_1, x_2, x_3 \in V_n$ are the corner vertices and $y_1, y_2, y_3 \in V_{n+1}\backslash V_n$ are the corresponding cut points (each y_i opposite x_i), the rule gives $h = h(x_1)\psi_1 + h(x_2)\psi_2 + h(x_3)\psi_3$ on this n-cell, where ψ_i is the unique harmonic spline with $\psi_i(x_j) = \delta_{ij}$, and the algorithm gives

$$\psi_1(y_1) = \frac{1}{5}, \quad \psi_1(y_2) = \psi_1(y_3) = \frac{2}{5}, \tag{1}$$

The other values of $\psi_i(y_j)$ are found via dihedral symmetry. Consequently h is discrete harmonic:

$$h(y_1) = \frac{1}{4}(h(x_2) + h(x_3) + h(y_2) + h(y_3)). \tag{2}$$

Moreover, since $\sum_{i=1}^3 \psi_i \equiv 1$ and $\mu(m\text{-cell}) = 3^{-m}$ for every $m \geq n$ we deduce by symmetry that for every $(n+1)$-subcell $\mathcal{T}_{n+1}$ contained in the n-cell, $\int_{\mathcal{T}_{n+1}} \psi_i d\mu = 3^{-(n+2)}$ for $i \in \{1, 2, 3\}$. We now establish the mean-value property of h over the ball $B(y_1, 2^{-(n+1)})$:

$$\frac{1}{\mu(B(y_1, 2^{-(n+1)}))} \int_{B(y_1,2^{-(n+1)})} h \, d\mu = h(y_1). \tag{3}$$

Using (1) and the above integrals of ψ_i, we get

$$\begin{aligned} \int_{B(y_1,2^{-(n+1)})} h \, d\mu &= \big(h(y_1) + h(x_3) + h(y_2)\big) \cdot 3^{-(n+2)} + \big(h(y_1) + h(x_2) + h(y_3)\big) \cdot 3^{-(n+2)} \\ &= \frac{1}{3} \cdot 3^{-(n+1)} \cdot \big(h(x_2) + h(x_3) + h(y_2) + h(y_3) + 2h(y_1)\big) \end{aligned}$$

$$= \frac{1}{3} \cdot 3^{-(n+1)} \cdot 6h(y_1),$$

where we used (2) in the last equation. Dividing the last expression by $\mu(B(y_1, 2^{-(n+1)})) = 2 \cdot 3^{-(n+1)}$ gives (3). In (3) we apply the self-similar cell structure of SG to deduce the claimed mean-value property. □

The next result relates the Green function on SG to the effective resistance. The definition of the effective resistance metric is standard, and can for example be found in Strichartz (2006, Section 1.6.).

Proposition 2.1 *Let $R_{eff}(A, B)$ be the effective resistance between subsets $A, B \subset SG$, subject to Dirichlet boundary conditions on V_0. Let $G(x, y)$ be the Green's function on SG subject to Dirichlet boundary conditions on V_0. Then the following holds:*

(1) *For every $x \in SG$, $G(x, x) = R_{eff}(x, V_0)$.*
(2) *For every $x, y \in SG$, $G(x, y) = \frac{1}{2}(G(x, x) + G(y, y) - R_{eff}(x, y))$.*
(3) *For every $x, y \in SG$ we have $G(x, x) \geq G(x, y)$.*

Proof These follow from the general theory of Green functions on resistance forms, which is detailed in Kigami (1015, Section 4). For proofs, see Theorems 4.1 and Theorem 4.3 from Kigami (1015). □

Lemma 2.3 *For every $x, y, z \in SG$ we have*

$$|G(x, y) - G(x, z)| \leq R_{eff}(y, z),$$

and consequently

$$0 \leq G(x, x) - G(x, y) \leq R_{eff}(x, y).$$

Proof Using (1) and (2) in Proposition 2.1, we obtain

$$\begin{aligned} G(x, y) - G(x, z) &= \frac{1}{2}[G(y, y) - G(z, z) + R_{eff}(x, z) - R_{eff}(x, y)] \\ &= \frac{1}{2}[R_{eff}(y, V_0) - R_{eff}(z, V_0) + R_{eff}(x, z) - R_{eff}(x, y)]. \end{aligned}$$

Note that V_0 is grounded and hence can be collapsed to a single vertex. Meanwhile, the resistance metric satisfies the triangle inequality, so

$$\begin{aligned} |G(x, y) - G(x, z)| &\leq \frac{1}{2}(|R_{eff}(y, V_0) - R_{eff}(z, V_0)| + |R_{eff}(x, z) - R_{eff}(x, y)|) \\ &\leq \frac{1}{2}(R_{eff}(y, z) + R_{eff}(y, z)) = R_{eff}(y, z), \end{aligned}$$

proving the first inequality. This together with (3) from Proposition 3 implies the second part. □

Lemma 2.4 *There exists a constant $C > 0$ such that the following hold.*

(1) *For every $x, y \in SG$ belonging to the same n-cell, we have*

$$0 \leq G(x, x) - G(x, y) \leq C \left(\frac{3}{5}\right)^n,$$

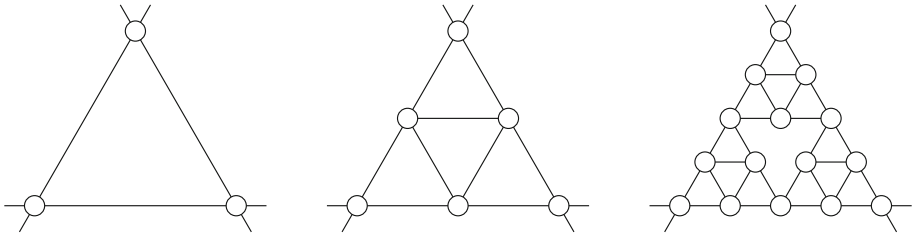


Fig. 2 The Sierpiński gasket graphs SG_0, SG_1 and SG_2 with normal boundary conditions

(2) *For every $x \in SG$, $n \in \mathbb{N}$ and $y \in SG$ with $d(y, V_0) \leq 2^{-n}$ we have*

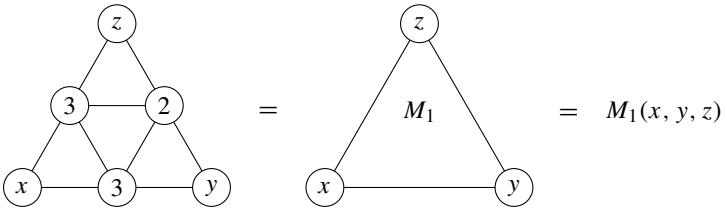
$$0 \leq G(x, y) \leq C \left(\frac{3}{5}\right)^n.$$

Proof For the first claim, using Lemma 2.3 it suffices to show that there exists $C > 0$ such that for every $x, y \in SG$ belonging to the same n-cell, we have $R_{eff}(x, y) \leq C \left(\frac{3}{5}\right)^n$. This is the well-known resistance upper bound (Strichartz 2006, Section 1.6). For the second claim, using the third and then the first item in Proposition 2.1, we obtain $G(x, y) \leq G(y, y) = R_{eff}(y, V_0)$. Since y shares the same n-cell with one of the corner vertices in V_0, applying the aforementioned resistance upper bound yields $R_{eff}(y, V_0) \leq C \left(\frac{3}{5}\right)^n$. □

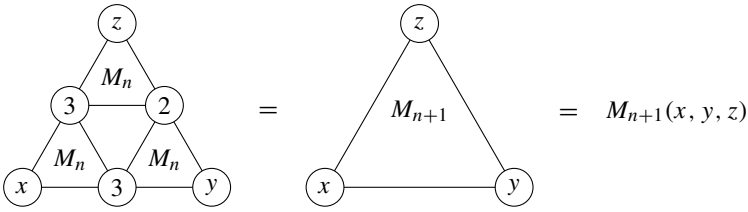
2.2 Sandpiles on the Sierpiński gasket graphs

Boundary conditions on SG_n. For the abelian sandpile Markov chain on SG_n, we adopt the standard "normal" boundary conditions used in the literature (e.g., Kaiser and Sava-Huss 2026): introduce a sink vertex $s \notin V_{SG_n}$ and connect it by two edges to the corner vertices of SG_n, so every vertex in V_{SG_n} has degree 4 (see Fig. 2). Throughout, "sandpile group on SG_n" refers to the set of recurrent configurations of the sandpile Markov chain with this boundary, denoted $\mathcal{R}_n$, and we write id_n for its identity element.

Sandpile identity on SG_n. Define the sandpile configuration $M_1(x, y, z)$ on SG_1 as



where the corner values $x, y, z \in \mathbb{N}$ are arbitrary. Iteratively define the configuration $M_{n+1}(x, y, z)$



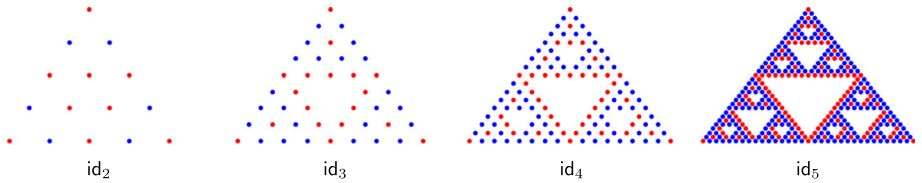
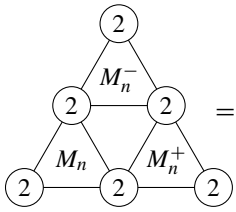


Fig. 3 The identity element id_n on SG_n for $n \in \{2, 3, 4, 5\}$. Blue dots are vertices with 3 chips and red dots are vertices with 2 chips (color figure online)

In Kaiser et al. (2024), the authors prove in Theorem 3.2 that for any $n \geq 1$, the identity element of SG_{n+1} with normal boundary conditions is given by



$= \mathsf{id}_{n+1},$

where M_n^+ (respectively M_n^-) denotes the sandpile configuration obtained from M_n by rotating SG_n counterclockwise (respectively clockwise) by 120°. The sandpile identity elements on the Sierpiński gaskets SG_n for $n \in \{2, 3, 4, 5\}$ are displayed in Fig. 3.

3 Decomposition of the sandpile identity

In this section, we establish a decomposition of the sandpile identity needed for proving the main theorem. For $n \in \mathbb{N}$, define $I_n : V_{SG_n} \to \mathbb{R}$ by

$$I_n(x) = (g_n * \mathsf{id}_n)(x) := \sum_{y \in V_{SG_n}} g_n(x, y)\, \mathsf{id}_n(y).$$

We have $\Delta_n I_n = \mathsf{id}_n$ on $V_{SG_n} \backslash \{u_1, u_2, u_3\}$ and $I_n(u_i) = 0$ for each $i \in \{1, 2, 3\}$; that is, I_n vanishes at the corner vertices of SG_n. We also define $h_n : V_{SG_n} \to \mathbb{R}$ by

$$h_n(x) = \sum_{y \in V_{SG_n}} g_n(x, y). \tag{4}$$

It follows that $\Delta_n h_n(x) = 1$ for all $x \in V_{SG_n} \backslash \{u_1, u_2, u_3\}$, and h_n vanishes at the corner vertices of SG_n. Next, define the distance to the corners by

$$d_n : V_{SG_n} \to \mathbb{R}, \quad d_n(x) = \min\big\{ d_{SG_n}(x, u_i) : i \in \{1, 2, 3\} \big\}, \tag{5}$$

where d_{SG_n} denotes the graph distance in SG_n. In the next proposition, we show that I_n can be expressed as a linear combination of the functions d_n and h_n.

Proposition 3.1 *For $n \geq 2$ and $I_n = g_n * \mathsf{id}_n$, it holds that*

$$I_n = -\frac{1}{3} d_n + \frac{8}{3} h_n.$$

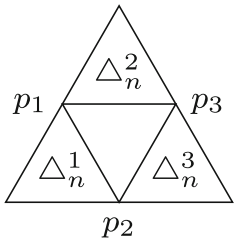


Fig. 4 Illustration of the cutpoints p_1, p_2, p_3 and the subtriangles $\triangle_n^i$ for $i \in \{1, 2, 3\}$ in SG_n

Proof We write $F_n := -\frac{1}{3}d_n + \frac{8}{3}h_n$ and we first show that

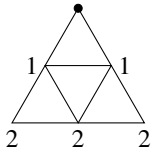
$$\Delta_n F_n(x) = \mathsf{id}_n(x) \quad \text{for all} \quad x \in V_{SG_n} \backslash \{u_1, u_2, u_3\}.$$

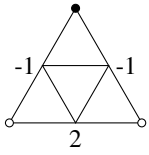
Since the functions d_n and h_n are rotationally symmetric, it suffices to consider $\Delta_n F_n$ only on the top triangle $\triangle_n^2$. Special attention must be paid to the cutpoints whose neighborhoods intersect more than one of the three subtriangles, since the Laplacian at a vertex depends on the values of the function on its neighboring vertices. We write $\triangle_n^2$ for the vertices of SG_n in the upper triangle $\triangle_n^2 = \psi_2(V_{SG_{n-1}} \backslash \{u_1, u_2, u_3\})$ and similarly

$$\triangle_n^1 = \psi_1\big(V_{SG_{n-1}} \backslash \{u_1, u_2, u_3\}\big), \quad \triangle_n^3 = \psi_3\big(V_{SG_{n-1}} \backslash \{u_1, u_2, u_3\}\big).$$

We write p_1, p_2 and p_3 for the three cutpoints, so $p_1 = \frac{1}{2}u_2$, $p_2 = \frac{1}{2}u_3$, $p_3 = p_1 + p_2$; see Fig. 4. Below, for readability reasons, we suppress the subscript n of the Laplacians in the calculations. We prove the statement by induction on the iteration of the gasket graphs; let $n = 2$.

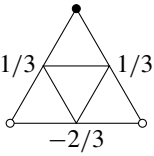
The graph distance from the top vertex u_2 and its Laplacian are given by

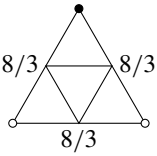
$d_2(x)|_{\triangle_2^2 \cup \{p_1, p_3\}} =$ 

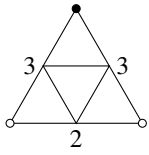
$\Delta d_2(x)|_{\triangle_2^2} =$ 

and the white circles placed at the cutpoints indicate values that we compute later. To emphasize that we work on $\triangle_2^2$, we have colored the top vertex u_2 black. Since the Laplacian of h_2 is 1 in all non-corner vertices, the Laplacian of F_2 on $\triangle_2^2$ can be computed as

$$\Delta F_2|_{\triangle_2^2} = -\tfrac{1}{3}\Delta d_2|_{\triangle_2^2} + \tfrac{8}{3}\Delta h_2|_{\triangle_2^2}$$

$=$ 

$+$ 

$=$ 

$= M_1^-.$

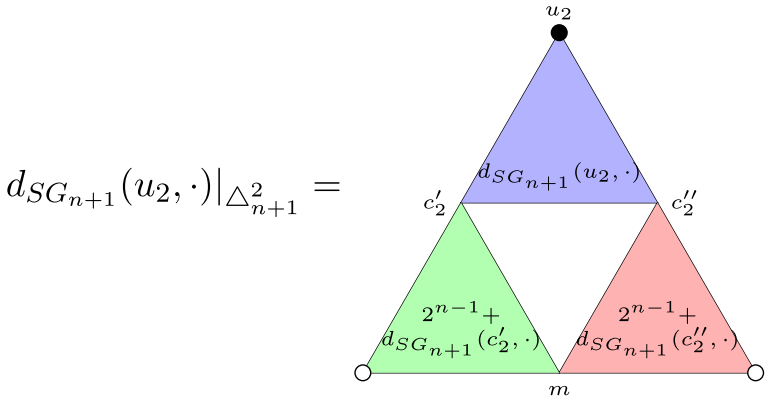


Fig. 5 The values of the graph distance function in the subtriangle $\triangle^2_{n+1}$

For the values at the cutpoints we have

$$\Delta F_2|_{\{p_1,p_2,p_3\}} = -\frac{1}{3}\Delta d_2|_{\{p_1,p_2,p_3\}} + \frac{8}{3}\Delta h_2|_{\{p_1,p_2,p_3\}}$$

$$= \quad -2/3 \quad -2/3 \quad -2/3 \quad + \quad 8/3 \quad 8/3 \quad 8/3 \quad = \quad 2 \quad 2 \quad 2$$

This proves the statement for $n = 2$.

Next we assume that $\Delta F_n|_{SG_n\setminus\{u_1,u_2,u_3\}} = \mathsf{id}_n|_{SG_n\setminus\{u_1,u_2,u_3\}}$ holds $n \in \mathbb{N}$, and we show that the statement holds for $n+1$. We restrict again our attention to the upper subtriangle $\triangle^2_{n+1}$. Within $\triangle^2_{n+1}$, the distance function d_{n+1} coincides with the graph distance to the top corner vertex u_2. Thus, on $\triangle^2_{n+1}$, the distance from the top corner can be depicted as in Fig. 5.

Since the Laplacian at x depends solely on the function's values at the neighbors of x, for all vertices in the blue triangle of Fig. 5 that have no neighbors in the red or green triangles, the claim follows directly from the induction hypothesis. Next, consider vertices lying in the green or red triangles that have no neighbors outside their respective triangles. Since within both the green and red regions the distance is uniformly shifted by the constant 2^{n-1}, the claim again follows from the induction hypothesis. Therefore, for any vertex x in the red, green, or blue triangle whose neighbors all lie in the same subtriangle as x, we have $\Delta F_{n+1}(x) = \mathsf{id}_{n+1}(x)$. Now we examine the values of the Laplacian at the cutpoints c_2', c_2'' and m in $\triangle^2_{n+1}$ as shown in Fig. 5. For c_2' we have

$$\begin{aligned}\Delta d_{SG_{n+1}}(u_2, c_2') &= \sum_{y\sim c_2'} d_{SG_{n+1}}(u_2, c_2') - d_{SG_{n+1}}(u_2, y)\\ &= 4\cdot 2^{n-1} - ((2^{n-1}-1) + 2^{n-1} + 2\cdot(2^{n-1}+1)) = -1.\end{aligned}$$

By symmetry, for c_2'' it holds that $\Delta d_{SG_{n+1}}(u_2, c_2'') = -1$, and for m similar computations lead to

$$\Delta d_{SG_{n+1}}(u_2, m) = 4\cdot 2^n - (2\cdot(2^n-1) + 2\cdot 2^n) = 2.$$

Hence, on $\triangle^2_{n+1}$ we have $\Delta F_{n+1}|_{\triangle^2_{n+1}} = \mathsf{id}_{n+1}|_{\triangle^2_{n+1}}$, and by rotational symmetry, the same holds on $\triangle^1_{n+1}$ and $\triangle^3_{n+1}$. It remains to address the cutpoints p_1, p_2, p_3 of SG_{n+1}; see Fig. 4 for the cutpoints of SG_n. By symmetry, the values of $\Delta d_{n+1}(\cdot)$ at these vertices agree and are equal to

$$\Delta d_{n+1}(p_1) = \sum_{v\sim p_1} d_{n+1}(p_1) - d_{n+1}(v) = 4\cdot 2^n - (2\cdot 2^n + 2\cdot(2^n-1)) = 2.$$

This completes the argument, yielding $\Delta_n F_n(x) = \mathsf{id}_n(x)$ for all $n \in \mathbb{N}$ and all $x \in V_{SG_n}\backslash\{u_1,u_2,u_3\}$. Define $T_n = F_n - I_n$. Then $\Delta_n T_n(x) = 0$ for all $x \in V_{SG_n}\backslash\{u_1,u_2,u_3\}$ and $T_n(u_i) = 0$ for each $i \in \{1,2,3\}$. By the maximum (–minimum) principle for harmonic functions, $T_n(x) = 0$ for all $x \in V_{SG_n}$, hence $I_n = F_n$, completing the proof. □

4 Scaling limit of the sandpile identity

We prove the three parts of Theorem 1.1 separately, beginning with (I2). This follows immediately from the decomposition of the sandpile identity established in Proposition 3.1.

Let $d_{SG}(x,y)$ denote the path distance between points $x, y \in SG$, which can be obtained from the graph distance on SG_n via

$$d_{SG}(x,y) = \lim_{n\to\infty} \frac{1}{2^n} d_{SG_n}(x_n, y_n),$$

where x_n and y_n are the vertices in SG_n closest to x and y respectively. Write

$$\mathsf{d}(x) = \min\{\, d_{SG}(x,u_1),\ d_{SG}(x,u_2),\ d_{SG}(x,u_3)\,\}.$$

Proposition 4.1 *It holds that*

$$\lim_{n\to\infty} \frac{1}{2^n}\left(g_n * \mathsf{id}_n - \frac{8}{3}h_n\right)^{\blacktriangle}(x) = -\frac{1}{3}\mathsf{d}(x).$$

Proof In view of Proposition 3.1

$$g_n * \mathsf{id}_n(x) - \frac{8}{3}h_n(x) = -\frac{1}{3}d_n(x),$$

for all $x \in V_{SG_n}$, from which the claim follows by using

$$\lim_{n\to\infty} \frac{1}{2^n}\left(g_n * \mathsf{id}_n - \frac{8}{3}h_n\right)^{\blacktriangle}(x) = \lim_{n\to\infty} -\frac{1}{3\cdot 2^n} d_n^{\blacktriangle}(x) = -\frac{1}{3}\mathsf{d}(x).$$

□

We now address statement (I3) of Theorem 1.1. The first step is to establish that the functions h_n provide a good approximation to the integral of the Green function G over the Sierpiński gasket.

Lemma 4.1 *There exists a constant $c_* > 0$ such that for all $x \in V_*$, it holds that*

$$\limsup_{n\to\infty} \left| 3\cdot 5^n \int_{SG} G(x,y)\,d\mu(y) - 2\sum_{y\in V_n} g_n(x,y) \right| \le c_*.$$

Proof First note that if $x \in V_0$, then $G(x, y) = 0$ for all $y \in SG$, and $g_n(x, y) = 0$ for all $n \in \mathbb{N}$ and all $y \in V_n$, making the lemma vacuously true.

Therefore we fix $x \in V_* \backslash V_0$. By definition, $x \in V_n$ for all sufficiently large $n \in \mathbb{N}$. Also, given $n \in \mathbb{N}$ we may partition SG into balls of diameter $2^{-(n+1)}$ in the Euclidean distance centered at each $z \in V_n$, $SG = \bigcup_{z \in V_n} B(z, 2^{-(n+1)})$. If $z \in V_n \backslash V_0$, then the closed ball $B(z, 2^{-(n+1)})$ consists of two $(n+1)$-cells adjoined at z, and if $z \in V_0$, then $B(z, 2^{-(n+1)})$ is the single $(n+1)$-cell containing the corner vertex z. As a result, under the standard self-similar probability measure μ, the mass assigned to each ball is

$$\mu\big(B(z, 2^{-(n+1)})\big) = \begin{cases} 2 \cdot 3^{-(n+1)}, & \text{if } z \in V_n \backslash V_0 \\ 3^{-(n+1)}, & \text{if } z \in V_0 \end{cases} \tag{6}$$

Note that while two different balls may intersect, the set of their intersection has μ-measure 0. We can thus express the integral of the Green's function as the sum of contributions from each ball of the said partition

$$\begin{aligned} \int_{SG} G(x, y) d\mu(y) &= \sum_{z \in V_n} \int_{B(z, 2^{-(n+1)})} G(x, y) d\mu(y) \\ &= \sum_{z \in V_n \backslash V_0} \int_{B(z, 2^{-(n+1)})} G(x, y) d\mu(y) + \sum_{z \in V_0} \int_{B(z, 2^{-(n+1)})} G(x, y) d\mu(y) \\ &=: I_{bulk} + I_{boundary}. \end{aligned}$$

To analyze I_{bulk}, by the harmonic spline construction of the Green's function, the function $G(x, \cdot)$ is harmonic on $B(z, 2^{-(n+1)})$ as long as $z \neq x$ and $z \notin V_0$. Applying Lemma 2.2, we get

$$\frac{1}{\mu(B(z, 2^{-(n+1)}))} \int_{B(z, 2^{-(n+1)})} G(x, y) d\mu(y) = G(x, z) \quad \text{for every } z \in V_n \backslash V_0,\ z \neq x. \tag{7}$$

which together with Eqs. (6) and (7) yields

$$\begin{aligned} I_{bulk} &= 2 \cdot 3^{-(n+1)} \sum_{z \in V_n \backslash V_0, z \neq x} G(x, z) + \int_{B(x, 2^{-(n+1)})} G(x, y) d\mu(y) \\ &= 2 \cdot 3^{-(n+1)} \sum_{z \in V_n} G(x, z) - 2 \cdot 3^{-(n+1)} G(x, x) + \int_{B(x, 2^{-(n+1)})} G(x, y) d\mu(y) \\ &= \frac{2}{3} \cdot \frac{1}{5^n} \sum_{z \in V_n} g_n(x, z) + \int_{B(x, 2^{-(n+1)})} \big(G(x, y) - G(x, x)\big) d\mu(y). \end{aligned}$$

Above, we used the fact that $G(x, z) = 0$ if one of x and z belongs to V_0, as well as Lemma 2.1 relating G to the discrete Green function. Turning now to the main expression in the Lemma, we can write

$$\begin{aligned} 3 \cdot 5^n \int_{SG} G(x, y) d\mu(y) - 2 \sum_{y \in V_n} g_n(x, y) &= 3 \cdot 5^n (I_{bulk} + I_{boundary}) - 2 \sum_{y \in V_n} g_n(x, y) \\ &= 3 \cdot 5^n \int_{B(x, 2^{-(n+1)})} \big(G(x, y) - G(x, x)\big) d\mu(y) + 3 \cdot 5^n \sum_{z \in V_0} \int_{B(z, 2^{-(n+1)})} G(x, y) d\mu(y). \end{aligned} \tag{8}$$

We claim that this expression is uniformly bounded in $n \in \mathbb{N}$ and in $x \in V_*$ by a universal constant. To prove this, we need two results concerning the Green's function, each of which

is used to bound each of the two terms in (8). The first result bounds the variation of the Green function on the balls $B(x, 2^{-(n+1)})$ for all $x \in V_* \backslash V_0$, $n \in \mathbb{N}$ and $y \in B(x, 2^{-(n+1)})$ and this is part (1) in Lemma 2.4, showing that the first term in Eq. (8) is negative. The second needed result bounds the decay of the Green function near the boundary V_0 and this is part (2) of Lemma 2.4, showing that the second term in (8) is positive. Therefore the absolute value of (8) is bounded above by

$$\begin{aligned}
&\max\left\{3\cdot 5^n \int_{B(x,2^{-(n+1)})} \big(G(x,x)-G(x,y)\big)d\mu(y),\ 3\cdot 5^n \sum_{z\in V_0}\int_{B(z,2^{-(n+1)})} G(x,y)d\mu(y)\right\}\\
&\le \max\left\{3\cdot 5^n\cdot 2\cdot 3^{-(n+1)}\cdot C\left(\frac{3}{5}\right)^{n+1},\ 3\cdot 5^n\cdot 3\cdot 3^{-(n+1)}\cdot C\left(\frac{3}{5}\right)^{n+1}\right\}\\
&= \max\left\{\frac{6}{5}C,\ \frac{9}{5}C\right\} = \frac{9}{5}C,
\end{aligned}$$

thereby proving the above claim and hence the Lemma. □

We now prove (I3) in Theorem 1.1.

Proposition 4.2 *For all $x \in V_*$ it holds that*

$$\lim_{n\to\infty}\frac{1}{2^n}\left(g_n * \mathsf{id}_n(x) - 4\cdot 5^n\int_{SG} G(x,y)d\mu(y)\right) = -\frac{1}{3}\mathsf{d}(x).$$

Proof Let $x \in V_*$, then by Proposition 3.1

$$\begin{aligned}
&\frac{1}{2^n}\left(g_n * \mathsf{id}_n(x) - 4\cdot 5^n\int_{SG} G(x,y)d\mu(y)\right)\\
&\quad = -\frac{1}{3\cdot 2^n}d_n(x) + \frac{4}{3\cdot 2^n}\left(2h_n(x) - 3\cdot 5^n\int_{SG} G(x,y)d\mu(y)\right).
\end{aligned}$$

Lemma 4.1 gives

$$\frac{4}{3\cdot 2^n}\left|2h_n(x) - 3\cdot 5^n\int_{SG} G(x,y)d\mu(y)\right| \le c_*\frac{4}{3\cdot 2^n}\xrightarrow{n\to\infty} 0,$$

which together with $\lim_n 2^{-n} d_{SG_n}(x) = \mathsf{d}(x)$ yield the claim. □

Using our previous result, we can now directly prove (I1) in Theorem 1.1 by using the already established (I3).

Proposition 4.3 *The function $g_n * \mathsf{id}_n$ grows asymptotically as 5^n, and for all $x \in SG$ it holds that*

$$\lim_{n\to\infty}\frac{1}{5^n}(g_n * \mathsf{id}_n)^{\blacktriangle}(x) = 4\int_{SG} G(x,y)d\mu(y).$$

Proof Recall for every $n \in \mathbb{N}$ the definition of the piecewise-constant continuatione of $u^{\blacktriangle} : SG \to \mathbb{R}$, $u^{\blacktriangle}(x) = u(x_n)$, where $x_n \in V_n$ is the closest point to $x \in SG$ in the n-th iteration. Let us introduce the shorthand notation $\beta_n : SG \to \mathbb{R}$ given by

$$\beta_n(x) := \frac{1}{5^n}(g_n * id_n)^{\blacktriangle}(x) - 4\int_{SG} G(x,y)d\mu(y),$$

which can be rewritten as

$$\begin{aligned}\beta_n(x) &= \left(\frac{1}{5^n}(g_n * id_n)^{\blacktriangle}(x) - 4\int_{SG}(G(\cdot,y))^{\blacktriangle}(x)d\mu(y)\right)\\ &\quad + 4\int_{SG}\big((G(\cdot,y))^{\blacktriangle}(x) - G(x,y)\big)d\mu(y)\\ &=: D_n(x) + D_n'(x).\end{aligned}$$

Using Proposition 4.2 and $\sup_{x\in SG}\mathsf{d}(x) = \frac{1}{2}$ we get

$$\begin{aligned}\limsup_{n\to\infty}|D_n(x)| &\le \limsup_{n\to\infty}\frac{1}{2^n}\left|(g_n * id_n)^{\blacktriangle}(x) - 4\cdot 5^n\int_{SG}(G(\cdot,y))^{\blacktriangle}(x)d\mu(y)\right|\\ &\quad\cdot\limsup_{n\to\infty}\frac{2^n}{5^n} = 0.\end{aligned}$$

On the other hand, by Lemma 2.3, for every $x \in SG$ we have

$$\big|(G(\cdot,y))^{\blacktriangle}(x) - G(x,y)\big| = |G(x_n,y) - G(x,y)| \le R_{eff}(x_n,x) \xrightarrow{n\to\infty} 0,$$

which together with the dominated convergence theorem yields $\lim_{n\to\infty} D_n'(x) = 0$. Hence it holds $\lim_{n\to\infty}\beta_n(x) = 0$ and this completes the proof. □

Remark 1 In this work, we examined the sequence of identity elements in the sandpile group on the Sierpiński gasket approximation graphs and showed that, after smoothing via convolution with the Green operator, the identity admits both first- and second-order scaling limits. The first-order limit is the integral of the Green function on the Sierpiński gasket, while the second-order limit is the distance to the gasket's corner vertices. An intriguing question is whether the emergence of the distance function in the Sierpiński gasket case is incidental and fails for other graph sequences, or whether a decomposition analogous to Proposition 3.1 also exists more broadly. Identifying such a structure for sandpile identities on more general graphs would allow to analyze scaling limits beyond the Sierpiński gasket, across a wider range of state spaces.

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References

Athreya SR, Járai AA (2004) Infinite volume limit for the stationary distribution of abelian sandpile models. Commun Math Phys 249(1):197–213

Bak P, Tang C, Wiesenfeld K (1987) Self-organized criticality: an explanation of the 1/f noise. Phys Rev Lett 59:381–384

Bak P, Tang C, Wiesenfeld K (1988) Self-organized criticality. Phys Rev A (3) 38(1):364–374

Barlow MT, Perkins EA (1988) Brownian motion on the Sierpiński gasket. Probab Theory Relat Fields 79(4):543–623

Bhupatiraju S, Hanson J, Járai AA (2017) Inequalities for critical exponents in d-dimensional sandpiles. Electron J Probab 22:85

Chen JP, Kudler-Flam J (2020) Laplacian growth and sandpiles on the Sierpiński gasket: limit shape universality and exact solutions. Ann Inst Henri Poincaré D 7(4):585–664

Dhar D (1990) Self-organized critical state of sandpile automaton models. Phys Rev Lett 64(14):1613–1616

Dhar D, Ruelle P, Sen S, Verma D-N (1995) Algebraic aspects of abelian sandpile models. J Phys A 28(4):805–831

Eckmann J-P, Nagnibeda T, Perriard A (2023) Abelian sandpiles on cylinders. J Phys A 56(17):175001

Fey A, Levine L, Peres Y (2010) Growth rates and explosions in sandpiles. J Stat Phys 138(1–3):143–159

Freiberg U, Heizmann N, Kaiser R, Sava-Huss E (2024) Internal aggregation models with multiple sources and obstacle problems on Sierpiński gaskets. J Fractal Geom 11(1–2):111–160

Hough RD, Jerison DC, Levine L (2019) Sandpiles on the square lattice. Commun Math Phys 367(1):33–87

Járai AA (2018) Sandpile models. Probab Surv 15:243–306

Járai AA, Lyons R (2007) Ladder sandpiles. Markov Process Relat Fields 13(3):493–518

Járai AA, Werning N (2014) Minimal configurations and sandpile measures. J Theor Probab 27(1):153–167

Jerison DC, Levine L, Pike J (2019) Mixing time and eigenvalues of the abelian sandpile Markov chain. Trans Am Math Soc 372(12):8307–8345

Kaiser R, Sava-Huss E (2026) Scaling limit of the sandpile identity element on the Sierpinski gasket. In: Ruiz PA, Hinz M, Okoudjou KA, Rogers LG, Teplyaev A (eds) From classical analysis to analysis on fractals: a tribute to Robert Strichartz. Applied and numerical harmonic analysis, vol 2. Birkhäuser

Kaiser R, Sava-Huss E, Wang Y (2024) Abelian sandpiles on Sierpiński gasket graphs. Electron J Comb 31(1):1.6

Kigami J (2001) Analysis on fractals, vol 143. Cambridge tracts in mathematics. Cambridge University Press, Cambridge

Kigami J (2012) Resistance forms, quasisymmetric maps and heat kernel estimates. Mem Am Math Soc 216(1015):vi+132

Le Borgne Y, Rossin D (2002) On the identity of the sandpile group, vol 256. In: LaCIM 2000 conference on combinatorics, computer science and applications (Montreal, QC), pp 775–790

Levine L, Pegden W, Smart CK (2016) Apollonian structure in the Abelian sandpile. Geom Funct Anal 26(1):306–336

Melchionna A (2022) The sandpile identity element on an ellipse. Discrete Contin Dyn Syst 42(8):3709–3732

Pegden W, Smart CK (2013) Convergence of the Abelian sandpile. Duke Math J 162(4):627–642

Strichartz RS (2006) Differential equations on fractals: a tutorial. Princeton University Press

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